A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we translate “unbroken curve” into a precise mathematical formulation called continuity, and develop some fundamental properties of continuous curves.

**DEFINITION OF CONTINUITY**

Intuitively, the graph of a function can be described as a “continuous curve” if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes. Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function \( f \) is undefined at \( c \) (Figure 1.5.1a).
- The limit of \( f(x) \) does not exist as \( x \) approaches \( c \) (Figures 1.5.1b, 1.5.1c).
- The value of the function and the value of the limit at \( c \) are different (Figure 1.5.1d).

This suggests the following definition.

1.5.1 **DEFINITION** A function \( f \) is said to be **continuous at** \( x = c \) provided the following conditions are satisfied:

1. \( f(c) \) is defined.
2. \( \lim_{x \to c} f(x) \) exists.
3. \( \lim_{x \to c} f(x) = f(c) \).

If one or more of the conditions of this definition fails to hold, then we will say that \( f \) has a **discontinuity at** \( x = c \). Each function drawn in Figure 1.5.1 illustrates a discontinuity at \( x = c \). In Figure 1.5.1a, the function is not defined at \( c \), violating the first condition of Definition 1.5.1. In Figure 1.5.1b, the one-sided limits of \( f(x) \) as \( x \) approaches \( c \) both exist but are not equal. Thus, \( \lim_{x \to c^-} f(x) \) does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1b has a **jump discontinuity** at \( c \). In Figure 1.5.1c, the one-sided limits of \( f(x) \) as \( x \) approaches \( c \) are infinite. Thus, \( \lim_{x \to c^-} f(x) \) does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1c has an **infinite discontinuity** at \( c \). In Figure 1.5.1d, the function is defined at \( c \) and \( \lim_{x \to c} f(x) \) exists, but these two values are not equal, violating the third condition of Definition 1.5.1. We will
say that a function like that in Figure 1.5.1 has a **removable discontinuity** at \( c \). Exercises 33 and 34 help to explain why discontinuities of this type are given this name.

### Example 1

Determine whether the following functions are continuous at \( x = 2 \).

- \( f(x) = \frac{x^2 - 4}{x - 2} \)
- \( g(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases} \)
- \( h(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases} \)

**Solution.** In each case we must determine whether the limit of the function as \( x \to 2 \) is the same as the value of the function at \( x = 2 \). In all three cases the functions are identical, except at \( x = 2 \), and hence all three have the same limit at \( x = 2 \), namely,

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4
\]

The function \( f \) is undefined at \( x = 2 \), and hence is not continuous at \( x = 2 \) (Figure 1.5.2a). The function \( g \) is defined at \( x = 2 \), but its value there is \( g(2) = 3 \), which is not the same as the limit as \( x \) approaches 2; hence, \( g \) is also not continuous at \( x = 2 \) (Figure 1.5.2b). The value of the function \( h \) at \( x = 2 \) is \( h(2) = 4 \), which is the same as the limit as \( x \) approaches 2; hence, \( h \) is continuous at \( x = 2 \) (Figure 1.5.2c). (Note that the function \( h \) could have been written more simply as \( h(x) = x + 2 \), but we wrote it in piecewise form to emphasize its relationship to \( f \) and \( g \).) 

### CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical events. For example, Figure 1.5.3a is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time \( t = t_0 \) (the voltage drops to zero when the line is cut). Figure 1.5.3b shows the graph of inventory versus time for a company that restocks its warehouse to \( y_1 \) units when the inventory falls to \( y_0 \) units. The discontinuities occur at those times when restocking occurs.
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**CONTINUITY ON AN INTERVAL**

If a function \( f \) is continuous at each number in an open interval \((a, b)\), then we say that \( f \) is continuous on \((a, b)\). This definition applies to infinite open intervals of the form \((a, +\infty)\), \((-\infty, b)\), and \((-\infty, +\infty)\). In the case where \( f \) is continuous on \((-\infty, +\infty)\), we will say that \( f \) is continuous everywhere.

Because Definition 1.5.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval \([a, b]\) or at the endpoint of an interval of the form \([a, b)\), \((a, b]\), \((-\infty, b]\), or \([a, +\infty)\). To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 1.5.4 is continuous at the right endpoint of the interval \([a, b]\) because

\[
\lim_{x \to b^-} f(x) = f(b)
\]

but it is not continuous at the left endpoint because

\[
\lim_{x \to a^+} f(x) \neq f(a)
\]

In general, we will say a function \( f \) is continuous from the left at \( c \) if

\[
\lim_{x \to c^-} f(x) = f(c)
\]

and is continuous from the right at \( c \) if

\[
\lim_{x \to c^+} f(x) = f(c)
\]

Using this terminology we define continuity on a closed interval as follows.

**1.5.2 Definition.** A function \( f \) is said to be continuous on a closed interval \([a, b]\) if the following conditions are satisfied:

1. \( f \) is continuous on \((a, b)\).
2. \( f \) is continuous from the right at \( a \).
3. \( f \) is continuous from the left at \( b \).

**Example 2** What can you say about the continuity of the function \( f(x) = \sqrt{9 - x^2} \)?

**Solution.** Because the natural domain of this function is the closed interval \([-3, 3]\), we will need to investigate the continuity of \( f \) on the open interval \((-3, 3)\) and at the two endpoints. If \( c \) is any point in the interval \((-3, 3)\), then it follows from Theorem 1.2.2(e) that

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)
\]

which proves \( f \) is continuous at each point in the interval \((-3, 3)\). The function \( f \) is also continuous at the endpoints since

\[
\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^-} (9 - x^2)} = 0 = f(-3)
\]

\[
\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)
\]

Thus, \( f \) is continuous on the closed interval \([-3, 3]\) (Figure 1.5.5).
1.5 Continuity

**SOME PROPERTIES OF CONTINUOUS FUNCTIONS**

The following theorem, which is a consequence of Theorem 1.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

1.5.3 **Theorem**  
If the functions \(f\) and \(g\) are continuous at \(c\), then

(a) \(f + g\) is continuous at \(c\).
(b) \(f - g\) is continuous at \(c\).
(c) \(fg\) is continuous at \(c\).
(d) \(f/g\) is continuous at \(c\) if \(g(c) \neq 0\) and has a discontinuity at \(c\) if \(g(c) = 0\).

We will prove part (d). The remaining proofs are similar and will be left to the exercises.

**Proof**  
First, consider the case where \(g(c) = 0\). In this case \(f(c)/g(c)\) is undefined, so the function \(f/g\) has a discontinuity at \(c\).

Next, consider the case where \(g(c) \neq 0\). To prove that \(f/g\) is continuous at \(c\), we must show that

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}
\]

Since \(f\) and \(g\) are continuous at \(c\),

\[
\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)
\]

Thus, by Theorem 1.2.2(d)

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}
\]

which proves (1).

**CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS**

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an arbitrary point. For example, we know from Theorem 1.2.3 that if \(p(x)\) is a polynomial and \(a\) is any real number, then

\[
\lim_{x \to a} p(x) = p(a)
\]

This shows that polynomials are continuous everywhere. Moreover, since rational functions are ratios of polynomials, it follows from part (d) of Theorem 1.5.3 that rational functions are continuous at points other than the zeros of the denominator, and at these zeros they have discontinuities. Thus, we have the following result.

1.5.4 **Theorem**

(a) A polynomial is continuous everywhere.
(b) A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.
Example 3
For what values of \( x \) is there a discontinuity in the graph of
\[ y = \frac{x^2 - 9}{x^2 - 5x + 6}. \]

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation
\[ x^2 - 5x + 6 = 0 \]
yields discontinuities at \( x = 2 \) and at \( x = 3 \) (Figure 1.5.6).

Example 4
Show that \( |x| \) is continuous everywhere (Figure 0.1.9).

Solution. We can write \( |x| \) as
\[ |x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases} \]
so \( |x| \) is the same as the polynomial \( x \) on the interval \((0, +\infty)\) and is the same as the polynomial \(-x\) on the interval \((-\infty, 0)\). But polynomials are continuous everywhere, so \( x = 0 \) is the only possible discontinuity for \( |x| \). Since \( |0| = 0 \), to prove the continuity at \( x = 0 \) we must show that
\[ \lim_{x \to 0} |x| = 0 \]  
(2)

Because the piecewise formula for \( |x| \) changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain
\[ \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0 \]
Thus, (2) holds and \( |x| \) is continuous at \( x = 0 \).

CONTINUITY OF COMPOSITIONS
The following theorem, whose proof is given in Appendix J, will be useful for calculating limits of compositions of functions.

1.5.5 Theorem
If \( \lim_{x \to c} g(x) = L \) and if the function \( f \) is continuous at \( L \), then
\[ \lim_{x \to c} f(g(x)) = f(L). \]
That is,
\[ \lim_{x \to c} f(g(x)) = f \left( \lim_{x \to c} g(x) \right) \]
This equality remains valid if \( \lim_{x \to c} g(x) \) is replaced everywhere by one of \( \lim_{x \to c^+}, \lim_{x \to c^-}, \lim_{x \to +\infty}, \) or \( \lim_{x \to -\infty} \).

In the special case of this theorem where \( f(x) = |x| \), the fact that \( |x| \) is continuous everywhere allows us to write
\[ \lim_{x \to c} |g(x)| = \left| \lim_{x \to c} g(x) \right| \]  
(3)
provided \( \lim_{x \to c} g(x) \) exists. Thus, for example,
\[ \lim_{x \to 3} |5 - x^2| = \left| \lim_{x \to 3} (5 - x^2) \right| = |-4| = 4 \]
The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific number and the second with continuity everywhere.

### 1.5.6 Theorem

(a) If the function \( g \) is continuous at \( c \), and the function \( f \) is continuous at \( g(c) \), then the composition \( f \circ g \) is continuous at \( c \).

(b) If the function \( g \) is continuous everywhere and the function \( f \) is continuous everywhere, then the composition \( f \circ g \) is continuous everywhere.

**Proof** We will prove part (a) only; the proof of part (b) can be obtained by applying part (a) at an arbitrary number \( c \). To prove that \( f \circ g \) is continuous at \( c \), we must show that the value of \( f \circ g \) and the value of its limit are the same at \( x = c \). But this is so, since we can write

\[
\lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f(g(x)) = f \left( \lim_{x \to c} g(x) \right) = f(g(c)) = (f \circ g)(c) \]

We know from Example 4 that the function \(|x|\) is continuous everywhere. Thus, if \( g(x) \) is continuous at \( c \), then by part (a) of Theorem 1.5.6, the function \(|g(x)|\) must also be continuous at \( c \); and, more generally, if \( g(x) \) is continuous everywhere, then so is \(|g(x)|\).

Stated informally:

*The absolute value of a continuous function is continuous.*

For example, the polynomial \( g(x) = 4 - x^2 \) is continuous everywhere, so we can conclude that the function \(|4 - x^2|\) is also continuous everywhere (Figure 1.5.7).

### The Intermediate-Value Theorem

Figure 1.5.8 shows the graph of a function that is continuous on the closed interval \([a, b]\). The figure suggests that if we draw any horizontal line \( y = k \), where \( k \) is between \( f(a) \) and \( f(b) \), then that line will cross the curve \( y = f(x) \) at least once over the interval \([a, b]\).

Stated in numerical terms, if \( f \) is continuous on \([a, b]\), then the function \( f \) must take on every value \( k \) between \( f(a) \) and \( f(b) \) at least once as \( x \) varies from \( a \) to \( b \). For example, the polynomial \( p(x) = x^5 - x + 3 \) has a value of 3 at \( x = 1 \) and a value of 33 at \( x = 2 \). Thus, it follows from the continuity of \( p \) that the equation \( x^5 - x + 3 = k \) has at least one solution in the interval \([1, 2]\) for every value of \( k \) between 3 and 33. This idea is stated more precisely in the following theorem.

### 1.5.7 Theorem (Intermediate-Value Theorem)

If \( f \) is continuous on a closed interval \([a, b]\) and \( k \) is any number between \( f(a) \) and \( f(b) \), inclusive, then there is at least one number \( x \) in the interval \([a, b]\) such that \( f(x) = k \).

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.
**1.5.8 Theorem** If \( f \) is continuous on \([a, b]\), and if \( f(a) \) and \( f(b) \) are nonzero and have opposite signs, then there is at least one solution of the equation \( f(x) = 0 \) in the interval \((a, b)\).

This result, which is illustrated in Figure 1.5.9, can be proved as follows.

**Proof** Since \( f(a) \) and \( f(b) \) have opposite signs, 0 is between \( f(a) \) and \( f(b) \). Thus, by the Intermediate-Value Theorem there is at least one number \( x \) in the interval \([a, b]\) such that \( f(x) = 0 \). However, \( f(a) \) and \( f(b) \) are nonzero, so \( x \) must lie in the interval \((a, b)\), which completes the proof.

Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If \( x \) is an approximation to a quantity \( x_0 \), then we call \( \epsilon = |x - x_0| \) the absolute error or (less precisely) the error in the approximation. The terminology in Table 1.5.1 is used to describe the size of such errors.

<table>
<thead>
<tr>
<th>Error</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>x - x_0</td>
</tr>
<tr>
<td>(</td>
<td>x - x_0</td>
</tr>
<tr>
<td>(</td>
<td>x - x_0</td>
</tr>
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<td>(</td>
<td>x - x_0</td>
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<td>(</td>
<td>x - x_0</td>
</tr>
<tr>
<td>(</td>
<td>x - x_0</td>
</tr>
<tr>
<td>(</td>
<td>x - x_0</td>
</tr>
</tbody>
</table>

**Example 5** The equation \( x^3 - x - 1 = 0 \) cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph \( p(x) = x^3 - x - 1 \) with a graphing utility (Figure 1.5.10), then we are led to conjecture that there is one real root and that this root lies inside the interval \([1, 2]\). The existence of a root in this interval is also confirmed by Theorem 1.5.8, since \( p(1) = -1 \) and \( p(2) = 5 \) have opposite signs. Approximate this root to two decimal-place accuracy.
1.5 Continuity

**Solution.** Our objective is to approximate the unknown root \( x_0 \) with an error of at most 0.005. It follows that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most \( \frac{1}{2}(0.01) = 0.005 \), which will achieve the desired accuracy.

We know that the root \( x_0 \) lies in the interval \([1, 2]\). However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval \([1, 2]\) into 10 equal parts and evaluating \( p \) at the points of subdivision using a calculating utility (Table 1.5.2). In this table \( p(1.3) \) and \( p(1.4) \) have opposite signs, so we know that the root lies in the interval \([1.3, 1.4]\). This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval \([1.3, 1.4]\) into 10 parts and evaluating \( p \) at the points of subdivision; this yields Table 1.5.3, which tells us that the root is inside the interval \([1.32, 1.33]\) (Figure 1.5.11). Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, \( x_0 \approx 1.325 \) to two decimal-place accuracy.

**Table 1.5.2**

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
<th>1.6</th>
<th>1.7</th>
<th>1.8</th>
<th>1.9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>-1</td>
<td>-0.77</td>
<td>-0.47</td>
<td>-0.10</td>
<td>0.34</td>
<td>0.88</td>
<td>1.50</td>
<td>2.21</td>
<td>3.03</td>
<td>3.96</td>
<td>5</td>
</tr>
</tbody>
</table>

**Table 1.5.3**

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.3</th>
<th>1.31</th>
<th>1.32</th>
<th>1.33</th>
<th>1.34</th>
<th>1.35</th>
<th>1.36</th>
<th>1.37</th>
<th>1.38</th>
<th>1.39</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>-0.103</td>
<td>-0.062</td>
<td>-0.020</td>
<td>0.023</td>
<td>0.066</td>
<td>0.110</td>
<td>0.155</td>
<td>0.201</td>
<td>0.248</td>
<td>0.296</td>
<td>0.344</td>
</tr>
</tbody>
</table>

**Remark.** To say that \( x \) approximates \( x_0 \) to \( n \) decimal places does not mean that the first \( n \) decimal places of \( x \) and \( x_0 \) will be the same when the numbers are rounded to \( n \) decimal places. For example, \( x = 1.084 \) approximates \( x_0 = 1.087 \) to two decimal places because \( |x - x_0| = 0.003 (< 0.005) \). However, if we round these values to two decimal places, then we obtain \( x \approx 1.08 \) and \( x_0 \approx 1.09 \). Thus, if you approximate a number to \( n \) decimal places, then you should display that approximation to at least \( n + 1 \) decimal places to preserve the accuracy.

**Quick Check Exercises 1.5** *(See page 120 for answers.)*

1. What three conditions are satisfied if \( f \) is continuous at \( x = c \)?
2. Suppose that \( f \) and \( g \) are continuous functions such that \( f(2) = 1 \) and \( \lim_{{x \to 2}} [f(x) + 4g(x)] = 13 \). Find
   (a) \( g(2) \)
   (b) \( \lim_{{x \to 2}} g(x) \).
3. Suppose that \( f \) and \( g \) are continuous functions such that \( \lim_{{x \to 3}} g(x) = 5 \) and \( f(3) = -2 \). Find \( \lim_{{x \to 3}} [f(x)/g(x)] \).
4. For what values of \( x \), if any, is the function \( f(x) = \frac{x^2 - 16}{x^3 - 5x + 4} \) discontinuous?
5. Suppose that a function \( f \) is continuous everywhere and that \( f(-2) = 3, f(-1) = -1, f(0) = -4, f(1) = 1, \) and \( f(2) = 5 \). Does the Intermediate-Value Theorem guarantee that \( f \) has a root on the following intervals?
   (a) \([-2, -1]\)  (b) \([-1, 0]\)  (c) \([-1, 1]\)  (d) \([0, 2]\)
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EXERCISE SET 1.5

Let \( f \) be the function whose graph is shown. On which of the following intervals, if any, is \( f \) continuous?

(a) \([1, 3]\)  
(b) \((1, 3)\)  
(c) \([1, 2]\)  
(d) \((1, 2)\)  
(e) \([2, 3]\)  
(f) \((2, 3)\)

For each interval on which \( f \) is not continuous, indicate which conditions for the continuity of \( f \) do not hold.

1. \[ \begin{align*} f(x) = 1, & \quad x \neq 4 \\ -1, & \quad x = 4 \end{align*} \]

2. \[ \begin{align*} g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases} \]

In each part, is the given function continuous at \( x = 4 \)?

(a) \( f(x) \)  
(b) \( g(x) \)  
(c) \(-g(x)\)  
(d) \( f(x) \)  
(e) \( f(x)g(x) \)  
(f) \( g(f(x)) \)  
(g) \( g(x) + 6f(x) \)

Consider the functions

3. \[ f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 0, & x < 0 \end{cases} \]

4. \[ g(x) = \begin{cases} 0, & 0 \leq x \\ 1, & x < 0 \end{cases} \]

In each part, is the given function continuous at \( x = 0 \)?

(a) \( f(x) \)  
(b) \( g(x) \)  
(c) \( f(-x) \)  
(d) \( |g(x)| \)  
(e) \( f(x)g(x) \)  
(f) \( g(f(x)) \)  
(g) \( f(x) + g(x) \)

Consider the functions

5. \[ f(x) = \begin{cases} x + 2, & x < 4 \\ 2x^2 + x, & x \geq 4 \end{cases} \]

6. \[ g(x) = \begin{cases} 3x - 1, & x \neq 3 \\ x - 1, & x = 3 \end{cases} \]

In each part, is the given function continuous at \( x = 0 \) or \( x = 3 \)?

(a) \( f(x) \)  
(b) \( g(x) \)  
(c) \( f(x) \)  
(d) \( g(x) \)  
(e) \( f(x)g(x) \)  
(f) \( g(f(x)) \)  
(g) \( f(x) + g(x) \)

In each part sketch the graph of a function \( f \) that satisfies the stated conditions.

(a) \( f \) is continuous everywhere except at \( x = 3 \), at which point it is continuous from the right.

(b) \( f \) has a two-sided limit at \( x = 3 \), but it is not continuous at \( x = 3 \).

(c) \( f \) is not continuous at \( x = 3 \), but if its value at \( x = 3 \) is changed from \( f(3) = 1 \) to \( f(3) = 0 \), it becomes continuous at \( x = 3 \).

(d) \( f \) is continuous on the interval \([0, 3]\) and is defined on the closed interval \([0, 3]\); but \( f \) is not continuous on the interval \([0, 3]\).

The accompanying figure models the concentration \( C \) of medication in the bloodstream of a patient over a 48-hour period of time. Discuss the significance of the discontinuities in the graph.

9. A student parking lot at a university charges $2.00 for the first half hour (or any part) and $1.00 for each subsequent half hour (or any part) up to a daily maximum of $10.00.

(a) Sketch a graph of cost as a function of the time parked.

(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.

(a) The Earth’s population as a function of time.

(b) Your exact height as a function of time.

(c) The cost of a taxi ride in your city as a function of the distance traveled.

(d) The volume of a melting ice cube as a function of time.

11–22 Find values of \( x \), if any, at which \( f \) is not continuous.

11. \( f(x) = 5x^4 - 3x + 7 \)

12. \( f(x) = \sqrt{x} - 8 \)

13. \( f(x) = \frac{x + 2}{x^2 + 4} \)

14. \( f(x) = \frac{x + 2}{x^2 - 4} \)

15. \( f(x) = \frac{x}{2x^2 + x} \)

16. \( f(x) = \frac{2x + 1}{4x^2 + 4x + 5} \)

17. \( f(x) = \frac{3 + x}{x^2 - 1} \)

18. \( f(x) = \frac{2x}{x + 4} \)

19. \( f(x) = \frac{x^2 + 6x + 9}{|x| + 3} \)

20. \( f(x) = 4 - \frac{8}{x^2 + x} \)

21. \( f(x) = \begin{cases} 2x + 3, & x \leq 4 \\ 7 + \frac{16}{x}, & x > 4 \end{cases} \)

22. \( f(x) = \begin{cases} 3, & x \neq 1 \\ x - 1, & x = 1 \end{cases} \)

23–28 True–False Determine whether the statement is true or false. Explain your answer.

23. If \( f(x) \) is continuous at \( x = c \), then so is \( |f(x)| \).

24. If \( |f(x)| \) is continuous at \( x = c \), then so is \( f(x) \).

25. If \( f \) and \( g \) are discontinuous at \( x = c \), then so is \( f + g \).

26. If \( f \) and \( g \) are discontinuous at \( x = c \), then so is \( fg \).
27. If \( \sqrt{f(x)} \) is continuous at \( x = c \), then so is \( f(x) \).
28. If \( f(x) \) is continuous at \( x = c \), then so is \( \sqrt{f(x)} \).

29–30 Find a value of the constant \( k \), if possible, that will make the function continuous everywhere.

29. (a) \( f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1 \end{cases} \)
   (b) \( f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x + k, & x > 2 \end{cases} \)

30. (a) \( f(x) = \begin{cases} 9 - x^2, & x \geq -3 \\ k/x^2, & x < -3 \end{cases} \)
   (b) \( f(x) = \begin{cases} 9 - x^2, & x \geq 0 \\ k/x^2, & x < 0 \end{cases} \)

31. Find values of the constants \( k \) and \( m \), if possible, that will make the function \( f \) continuous everywhere.
   \[ f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x + 1) + k, & -1 < x \leq 2 \\ 2x^3 + x + 7, & x \leq -1 \end{cases} \]

32. On which of the following intervals is \( f(x) = \frac{1}{\sqrt{x - 2}} \) continuous?
   (a) \( [2, +\infty) \) (b) \( (-\infty, +\infty) \) (c) \( (2, +\infty) \) (d) \( [1, 2) \)

33–36 A function \( f \) is said to have a removable discontinuity at \( x = c \) if \( \lim_{x \to c} f(x) \) exists but \( f(x) \) is not continuous at \( x = c \), either because \( f \) is not defined at \( c \) or because the definition for \( f(c) \) differs from the value of the limit. This terminology will be needed in these exercises.

33. (a) Sketch the graph of a function with a removable discontinuity at \( x = c \) for which \( f(c) \) is undefined.
   (b) Sketch the graph of a function with a removable discontinuity at \( x = c \) for which \( f(c) \) is defined.

34. (a) The terminology removable discontinuity is appropriate because a removable discontinuity of a function \( f \) at \( x = c \) can be “removed” by redefining the value of \( f \) appropriately at \( x = c \). What value for \( f(c) \) removes the discontinuity?
   (b) Show that the following functions have removable discontinuities at \( x = 1 \), and sketch their graphs.
   \[ f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases} \]
   (c) What values should be assigned to \( f(1) \) and \( g(1) \) to remove the discontinuities?

35–36 Find the values of \( x \) (if any) at which \( f \) is not continuous, and determine whether each such value is a removable discontinuity.

35. (a) \( f(x) = \frac{|x|}{x} \) (b) \( f(x) = \frac{x^2 + 3x}{x + 3} \)
   (c) \( f(x) = \frac{x - 2}{|x| - 2} \)

36. (a) \( f(x) = \frac{x^2 - 4}{x^3 - 8} \) (b) \( f(x) = \begin{cases} 2x - 3, & x \leq 2 \\ x^2, & x > 2 \end{cases} \)
   (c) \( f(x) = \begin{cases} 3x^2 + 5, & x \neq 1 \\ 6, & x = 1 \end{cases} \)

37. (a) Use a graphing utility to generate the graph of the function \( f(x) = (x + 3)/(2x^2 + 5x - 3) \), and then use the graph to make a conjecture about the number and locations of all discontinuities.
   (b) Check your conjecture by factoring the denominator.

38. (a) Use a graphing utility to generate the graph of the function \( f(x) = x/(x^3 - x + 2) \), and then use the graph to make a conjecture about the number and locations of all discontinuities.
   (b) Use the Intermediate-Value Theorem to approximate the locations of all discontinuities to two decimal places.

39. Prove that \( f(x) = x^{3/5} \) is continuous everywhere, carefully justifying each step.
40. Prove that \( f(x) = 1/\sqrt{x^4 + 7x^3 + 1} \) is continuous everywhere, carefully justifying each step.

41. Prove:
   (a) part (a) of Theorem 1.5.3
   (b) part (b) of Theorem 1.5.3
   (c) part (c) of Theorem 1.5.3.

42. Prove part (b) of Theorem 1.5.4.

43. (a) Use Theorem 1.5.5 to prove that if \( f \) is continuous at \( x = c \), then \( \lim_{h \to 0} f(c + h) = f(c) \).
   (b) Prove that if \( \lim_{h \to 0} f(c + h) = f(c) \), then \( f \) is continuous at \( x = c \). [Hint: What does this limit tell you about the continuity of \( g(h) = f(c + h) \)?]
   (c) Conclude from parts (a) and (b) that \( f \) is continuous at \( x = c \) if and only if \( \lim_{h \to 0} f(c + h) = f(c) \).

44. Prove: If \( f \) and \( g \) are continuous on \([a, b]\), and \( f(a) > g(a) \), \( f(b) < g(b) \), then there is at least one solution of the equation \( f(x) = g(x) \) in \((a, b)\). [Hint: Consider \( f(x) - g(x) \).]

**FOCUS ON CONCEPTS**

45. Give an example of a function \( f \) that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation \( f(x) = 0 \) has no solution in the interval.
46. Let \( f \) be the function whose graph is shown in Exercise 2. For each interval, determine (i) whether the hypothesis of the Intermediate-Value Theorem is satisfied, and (ii) whether the conclusion of the Intermediate-Value Theorem is satisfied.
   (a) \([1, 2]\)   (b) \([2, 3]\)   (c) \([1, 3]\)
47. Show that the equation \( x^3 + x^2 - 2x = 1 \) has at least one solution in the interval \([-1, 1]\).
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48. Prove: If \( p(x) \) is a polynomial of odd degree, then the equation \( p(x) = 0 \) has at least one real solution.

49. The accompanying figure shows the graph of the equation \( y = x^4 + x - 1 \). Use the method of Example 5 to approximate the \( x \)-intercepts with an error of at most 0.05.

50. The accompanying figure shows the graph of the equation \( y = 5 - x - x^4 \). Use the method of Example 5 to approximate the roots of the equation \( 5 - x - x^4 = 0 \) to two decimal-place accuracy.

51. Use the fact that \( \sqrt{5} \) is a solution of \( x^2 - 5 = 0 \) to approximate \( \sqrt{5} \) with an error of at most 0.005.

52. A sprinter, who is timed with a stopwatch, runs a hundred yard dash in 10 s. The stopwatch is reset to 0, and the sprinter is timed jogging back to the starting block. Show that there is at least one point on the track at which the reading on the stopwatch during the sprint is the same as the reading during the return jog. [Hint: Use the result in Exercise 44.]

53. Prove that there exist points on opposite sides of the equator that are at the same temperature. [Hint: Consider the accompanying figure, which shows a view of the equator from a point above the North Pole. Assume that the temperature \( T(\theta) \) is a continuous function of the angle \( \theta \), and consider the function \( f(\theta) = T(\theta + \pi) - T(\theta) \).]

✔️ QUICK CHECK ANSWERS 1.5

1. \( f(c) \) is defined; \( \lim_{x \to c} f(x) \) exists; \( \lim_{x \to c} f(x) = f(c) \)
2. (a) 3 (b) 3
3. \(-2/5\)
4. \( x = 1, 4 \)
5. (a) yes (b) no (c) yes (d) yes

54. Let \( R \) denote an elliptical region in the \( xy \)-plane, and define \( f(z) \) to be the area within \( R \) that is on, or to the left of, the vertical line \( x = z \). Prove that \( f \) is a continuous function of \( z \). [Hint: Assume the ellipse is between the horizontal lines \( y = a \) and \( y = b \), \( a < b \). Argue that \( |f(z_1) - f(z_2)| \leq (b - a) \cdot |z_1 - z_2| \).]

55. Let \( R \) denote an elliptical region in the plane. For any line \( L \), prove there is a line perpendicular to \( L \) that divides \( R \) in half by area. [Hint: Introduce coordinates so that \( L \) is the \( x \)-axis. Use the result in Exercise 54 and the Intermediate-Value Theorem.]

56. Suppose that \( f \) is continuous on the interval \([0, 1]\) and that \( 0 \leq f(x) \leq 1 \) for all \( x \) in this interval.
(a) Sketch the graph of \( y = x \) together with a possible graph for \( f \) over the interval \([0, 1]\).
(b) Use the Intermediate-Value Theorem to help prove that there is at least one number \( c \) in the interval \([0, 1]\) such that \( f(c) = c \).

57. Writing It is often assumed that changing physical quantities such as the height of a falling object or the weight of a melting snowball, are continuous functions of time. Use specific examples to discuss the merits of this assumption.

58. Writing The Intermediate-Value Theorem (Theorem 1.5.7) is an example of what is known as an “existence theorem.” In your own words, describe how to recognize an existence theorem, and discuss some of the ways in which an existence theorem can be useful.